

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

At the start of this year I stated (to numerous mathematician's and actuaries) that **power series** [Maclaurin and Taylor] and **Fourier series**, and consequently Fourier's Theorem, are particular applications of the diffusion equation, i.e., particular applications of the calculus extended from  $R$  to  $S$  for the case of  $R$ , and hence therefore extending these most useful and prominent techniques of mathematics to variables on  $S$  [those with stochastic variation, i.e., entropic variables] as made possible by extending the calculus from  $R$  to  $S$  with the diffusion equation.

Here I show the particular case of power series and then in the next document I show this for Fourier's theorem. This then allows these techniques to be extended also from  $R$  to  $S$ , as calculus dynamics, and proves definitively the power of the diffusion equation and it's all encompassing applications thereby proving my statements regarding 'all things between Newton and everything'. The calculus extended from  $R$  to  $S$ , via the diffusion equation, is the molecular building kit of dynamics and therefore as applicable in building up dynamics as tetrahedral carbon is for building up molecules. Fourier's theorem then shows definitively that long chains of harmonic roots in terms of the diffusion equation is as prevalent as long chain organic molecules [more than 14 million known as compiled in Beilstein references (Handbuch der Organischen Chemie)]. It is possible that there are diffusion equations with literally billions of roots, such as that expected from DNA, and undoubtedly as diverse as the organic compounds compiled in Beilstein. The Universe is dynamic and the calculus extended from  $R$  to  $S$  is synonymous with this dynamics, that of heat and diffusion [the diffusion equation as calculus on  $S$  and the heat equation as the probability of this dynamics]. I have good reason to believe that the diffusion equation will unify all of mathematics, as well as mathematical physics [Hilbert problem number 6] and chemistry, and engineering (on  $R$  for civil and mechanical, and on  $S$  for electrical), and this gives me confidence that it is possible to resolve all things.

### Here we go:

Remembering that the diffusion equation relates the diffusion variable  $X$  [the statistical data when on  $S$ ], that which is measured, with a [unobservable] basis variable  $z$  which generates the space on which the calculus operates

$$X_t = z_t$$

and to which the calculus dynamics is added. The parameter of dynamics is time  $t$ . This is the independent variable of Newton's Laws of Motion and that of time series of statistics, and that variable by which the future unfolds [such as frames per second in film]. In the case here, we will be considering all unit roots which hence do not have harmonics; adding  $n$  unit roots  $p(D) = D^n$  to the diffusion side [with no inverse diffusion  $q(D) = D^0 = 1$ ], the diffusion equation becomes

$$D^n X_t = z_t \qquad \left( \frac{d}{dt} \right)^n X_t = z_t \qquad \frac{d^n}{dt^n} X_t = z_t$$

Resolving

$$X_t = D^{-n} z_t \\ = \int_{-\infty}^{t_n} \int_{-\infty}^{t_{n-1}} \dots \int_{-\infty}^{t_1} z_t dt_1 \dots dt_{n-1} dt_n$$

which can be expressed as

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

(where  $t = t_1$ ) which is the formula for the resolved form of the diffusion equation for an arbitrary order  $n$  of repeated unit roots. This is the main formula of this document [from technique of BOOK3 of my PhD] which is utilized to prove my assertion that power series is an example of the diffusion equation.

### proof

For the trivial case of a single unit root (simple Brownian motion), the resolved equation is obtained by inverting the unit differential root into integral form

$$X_t = D^{-1} z_t = \int_{-\infty}^t z_y dy$$

For two unit roots

# Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

$$\begin{aligned}
 X_t = D^{-2} z_t &= \int_{-\infty}^t \int_{-\infty}^u z_y dy du = \int_{-\infty}^t \int_y^t z_y du dy = \int_{-\infty}^t \left( \int_y^t du \right) z_y dy = \int_{-\infty}^t (t-y) z_y dy \\
 &= \int_{-\infty}^t \frac{(t-y)}{\Gamma(2)} z_y dy
 \end{aligned}
 \tag{submitted in 2011}$$

and one more; for three unit roots

$$\begin{aligned}
 X_t = D^{-3} z_t &= \int_{-\infty}^t \int_{-\infty}^v \int_{-\infty}^u z_y dy du dv = \int_{-\infty}^t \int_y^t \int_u^t z_y dv du dy = \int_{-\infty}^t \left( \int_y^t \int_u^t dv du \right) z_y dy \\
 &= \int_{-\infty}^t \left( \int_y^t (t-u) du \right) z_y dy = \int_{-\infty}^t \left( \frac{1}{2} (t-u)^2 \right) z_y dy \\
 &= \int_{-\infty}^t \frac{(t-u)^2}{\Gamma(3)} z_y dy
 \end{aligned}$$

giving the starting pattern which is clear will generated powers with factorials in a repeated manner by repeated inner integration. Proving now with induction: assuming that it is true for  $n$

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

and is true for  $m$ , then it is proved that it is true if it is also shown to be true for  $n + m$ . The integral operator to resolve  $D^n$  is

$$D^{-n} = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} (\cdot) dy \text{ and that for } D^m \text{ is } D^{-m} = \int_{-\infty}^t \frac{(t-y)^{m-1}}{\Gamma(m)} (\cdot) dy.$$

Hence resolving the first  $n$  roots in  $p(D) = D^{n+m}$  for the diffusion equation with  $n+m$  unit roots  $D^{n+m} X_t = z_t$  with the  $n^{\text{th}}$  order integral  $D^{-n}$  giving  $D^m X_t = D^{-n} z_t$

$$D^m X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

and then next  $m$  roots together with the  $m^{\text{th}}$  order integral  $D^{-m}$  for a fully resolved variable  $X_t = D^{-m-n} z_t$

$$X_t = D^{-m} (D^{-n} z_t) = D^{-m} \left( \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \right) = \int_{-\infty}^t \frac{(t-y)^{m-1}}{\Gamma(m)} \left( \int_{-\infty}^y \frac{(y-u)^{n-1}}{\Gamma(n)} z_u du \right) dy$$

theory for which is from my Finance PhD thesis [submitted in 2011 and (more than) inappropriately failed and then defrauded by the idiot examiner] and as per theory from BOOK3 of my PhD, reversing the order of integration

$$X_t = \int_{-\infty}^t z_u \left( \int_u^t \frac{(t-y)^{m-1}}{\Gamma(m)} \frac{(y-u)^{n-1}}{\Gamma(n)} dy \right) du$$

The inner integral is then a beta integral. With change of variable  $w = (t-y)/(t-u)$

$$\begin{aligned}
 \int_u^t \frac{(t-y)^{m-1}}{\Gamma(m)} \frac{(y-u)^{n-1}}{\Gamma(n)} dy &= \frac{(t-u)^{n+m-2}}{\Gamma(n)\Gamma(m)} \int_u^t \left( \frac{t-y}{t-u} \right)^{m-1} \left( \frac{y-u}{t-u} \right)^{n-1} dy \\
 &= \frac{(t-u)^{n+m-1}}{\Gamma(n)\Gamma(m)} \int_0^1 w^{m-1} (1-w)^{n-1} dw \\
 &= \frac{(t-u)^{n+m-1}}{\Gamma(n)\Gamma(m)} \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \\
 &= \frac{(t-u)^{n+m-1}}{\Gamma(n+m)}
 \end{aligned}$$

so that

$$X_t = \int_{-\infty}^t \frac{(t-u)^{n+m-1}}{\Gamma(n+m)} z_u du$$

which is the fully resolved diffusion equation for  $D^{n+m}X_t = z_t$  proving that the result for all  $n$  and  $m$  [a proof also in a document that I sent to the maths people and actuaries a few years ago... time flies]...

Hence, the resolution of  $D^n X_t = z_t$  is

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

for any  $n$  proving the result for arbitrary order of repeated unit roots.

### end of proof

For the power series, I will again start out simply for  $n = 1, 2, 3, 4$ , and  $5$  and then for arbitrary  $n$  so the mathematical pattern of the power series can be seen clearly and made intuitive. This shows definitively that this is how Taylor and Maclaurin series arise, by reconstructing dynamics at the backward variables, and shows how techniques that utilize these expansions can extend them from  $R$  to  $S$ . For  $n = 1, 2, 3, 4$ , and  $5$  and then for arbitrary  $n$  is the approach I took in doing the mathematics on paper. On a programmable calculator for example, each pixel in an expansion will then be a variate from a normal [Gaussian] bell determined from the remainder term. The remainder term comprises the integrated basis [and hence contains  $\sigma$  if this is non-negligible with reference to  $\mu$ ] for the forward transition on  $S$  to the forward time variable  $t$ . This is the remainder term in the context of the diffusion equation and is the forward transition in probability into the uncertain future. This term, the remainder, vanishes as  $n \rightarrow \infty$ ].

For  $n=1$ , the diffusion equation obtained for one unit root  $DX_t = z_t$  is that of Browning motion which resolves for

$$\begin{aligned} X_t &= \int_{-\infty}^t z_y dy \\ &= \int_{-\infty}^0 z_y dy + \int_0^t z_y dy \\ &= D^{-1}z_0 + \int_0^t z_y dy \\ &= x_0 + \int_0^t z_y dy \end{aligned}$$

which is the somewhat trivial start to the series; a single term of power series  $x_0$  with remainder  $\int_0^t z_y dy$ .

For  $n=2$ , two unit roots the diffusion equation is  $D^2 X_t = z_t$  which is that of Integrated Browning motion. Resolving gives

$$\begin{aligned} X_t &= \int_{-\infty}^t \frac{(t-y)}{\Gamma(2)} z_y dy \\ &= \int_{-\infty}^0 \frac{(t-y)}{\Gamma(2)} z_y dy + \int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy \\ &= \frac{t}{\Gamma(2)} \int_{-\infty}^0 z_y dy + \frac{\Gamma(1)}{\Gamma(2)} \int_{-\infty}^0 \frac{(-y)}{\Gamma(1)} z_y dy + \int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy \\ &= \frac{t}{\Gamma(2)} D^{-1}z_0 + \frac{\Gamma(1)}{\Gamma(2)} D^{-2}z_0 + \int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy \\ &= t D x_0 + x_0 + \int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy \\ &= t \frac{dx_0}{dt} + x_0 + \int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy \end{aligned}$$

$$\begin{aligned} D^2 X_t &= z_t \\ D^1 X_t &= D^{-1} z_t \\ X_t &= D^{-2} z_t \\ \text{so} \\ D^{-1} z_0 &= D^1 x_0 \\ D^{-2} z_0 &= x_0 \end{aligned}$$

where the power series is  $(tD+1)x_0$  with remainder  $\int_0^t \frac{(t-y)}{\Gamma(2)} z_y dy$ . Note that the power series plus the remainder is the resolved form of Newton's Law of Motion for constant acceleration. In this case, the basis function is high SNR [Signal2Noise (Ratio),  $\mu/\sigma^2 \gg 1$ ] so that the basis variance is negligible and the remainder term evaluates to  $\frac{1}{2}\mu t^2$  (the Lagrange form of the remainder) giving the resolved variable for position/displacement, as an integration of velocity and double integration of acceleration, as

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

$X_t = x_0 + tDx_0 + \frac{1}{2}\mu t^2$  where  $Dx_0 = \dot{x}$  ( $x$  dot) is velocity and  $\mu = a = \ddot{x}$  ( $x$  double dot) is the [constant] acceleration and the basis mean (for  $\sigma^2 \approx 0$  as for trajectories of planets and cannon balls but not for electrons [and finance] for which  $\sigma^2 > 0$ ).

For  $n=3$ , three unit roots and hence the integration of integrated Brownian motion  $D^3X_t = z_t$

$$\begin{aligned}
 X_t &= \int_{-\infty}^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \int_{-\infty}^0 \frac{(t-y)^2}{\Gamma(3)} z_y dy + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \int_{-\infty}^0 \frac{t^2 + 2t(-y) + (-y)^2}{\Gamma(3)} z_y dy + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \frac{t^2}{\Gamma(3)} \int_{-\infty}^0 z_y dy + \frac{2t\Gamma(2)}{\Gamma(3)} \int_{-\infty}^0 \frac{(-y)}{\Gamma(2)} z_y dy + \frac{\Gamma(3)}{\Gamma(3)} \int_{-\infty}^0 \frac{(-y)^2}{\Gamma(3)} z_y dy + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \frac{t^2}{\Gamma(3)} D^{-1}z_0 + \frac{2t\Gamma(2)}{\Gamma(3)} D^{-2}z_0 + D^{-3}z_0 + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \frac{t^2}{2!} D^2x_0 + \frac{t}{1!} D^1x_0 + x_0 + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy \\
 &= \frac{t^2}{2!} \frac{d^2x_0}{dt^2} + t \frac{dx_0}{dt} + x_0 + \int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy
 \end{aligned}$$

$$\begin{aligned}
 D^3X_t &= z_t \\
 D^2X_t &= D^{-1}z_t \\
 D^1X_t &= D^{-2}z_t \\
 X_t &= D^{-3}z_t
 \end{aligned}$$

so

$$\begin{aligned}
 D^{-1}z_0 &= D^2x_0 \\
 D^{-2}z_0 &= D^1x_0 \\
 D^{-3}z_0 &= x_0
 \end{aligned}$$

where the power series is  $\left(\frac{1}{2}t^2D^2 + tD + 1\right)x_0$  with remainder  $\int_0^t \frac{(t-y)^2}{\Gamma(3)} z_y dy$ .

For  $n=4$ , four unit roots  $D^4X_t = z_t$  resolving each root in turn  $D^3X_t = D^{-1}z_t$ ,  $D^2X_t = D^{-2}z_t$ ,  $DX_t = D^{-3}z_t$ , and  $X_t = D^{-4}z_t$ , that is,

$$\begin{aligned}
 X_t &= \int_{-\infty}^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \int_{-\infty}^0 \frac{(t-y)^3}{\Gamma(4)} z_y dy + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \int_{-\infty}^0 \frac{t^3 + 3t^2(-y) + 3t(-y)^2 + (-y)^3}{\Gamma(4)} z_y dy + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \frac{t^3}{\Gamma(4)} \int_{-\infty}^0 z_y dy + \frac{3t^2\Gamma(2)}{\Gamma(4)} \int_{-\infty}^0 \frac{(-y)}{\Gamma(2)} z_y dy + \frac{3t\Gamma(3)}{\Gamma(4)} \int_{-\infty}^0 \frac{(-y)^2}{\Gamma(3)} z_y dy + \int_{-\infty}^0 \frac{(-y)^3}{\Gamma(4)} z_y dy + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \frac{t^3}{3!} D^{-1}z_0 + \frac{t^2}{2!} D^{-2}z_0 + tD^{-3}z_0 + D^{-4}z_0 + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \frac{t^3}{3!} D^3x_0 + \frac{t^2}{2!} D^2x_0 + tD^1x_0 + x_0 + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \\
 &= \frac{t^3}{3!} \frac{d^3x_0}{dt^3} + \frac{t^2}{2!} \frac{d^2x_0}{dt^2} + t \frac{dx_0}{dt} + x_0 + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy
 \end{aligned}$$

$$\begin{aligned}
 D^{-1}z_0 &= D^3x_0 & D^4X_t &= z_t \\
 D^{-2}z_0 &= D^2x_0 & D^3X_t &= D^{-1}z_t \\
 D^{-3}z_0 &= D^1x_0 & D^2X_t &= D^{-2}z_t \\
 D^{-4}z_0 &= x_0 & D^1X_t &= D^{-3}z_t \\
 & & X_t &= D^{-4}z_t
 \end{aligned}$$

where

for which it can be seen that the forward memory [which here is extended Markov to order 4] is a Maclaurin series,

$$\left(\frac{1}{3!}t^3D^3 + \frac{1}{2!}t^2D^2 + \frac{1}{1!}t^1D + 1\right)x_0$$

and the term for the forward transition is the remainder

$$\int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy$$

One more, to show clearly how the gamma factorials cancel from the binomial expansion, then for arbitrary  $n$ :

For  $n=5$   $D^5 X_t = z_t$ , resolving each root in turn  $D^4 X_t = D^{-1} z_t$ ,  $D^3 X_t = D^{-2} z_t$ ,  $D^2 X_t = D^{-3} z_t$ ,  $DX_t = D^{-4} z_t$ , and  $X_t = D^{-5} z_t$ , that is,

$$\begin{aligned} X_t &= \int_{-\infty}^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \int_{-\infty}^0 \frac{(t-y)^4}{\Gamma(5)} z_y dy + \int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \int_{-\infty}^0 \frac{t^4 + 4t^3(-y) + 6t^2(-y)^2 + 4t(-y)^3 + (-y)^4}{\Gamma(5)} z_y dy + \int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \frac{t^4}{\Gamma(5)} \int_{-\infty}^0 z_y dy + \frac{4t^3 \Gamma(2)}{\Gamma(5)} \int_{-\infty}^0 \frac{(-y)}{\Gamma(2)} z_y dy + \frac{6t^2 \Gamma(3)}{\Gamma(5)} \int_{-\infty}^0 \frac{(-y)^2}{\Gamma(3)} z_y dy + \frac{4t \Gamma(4)}{\Gamma(5)} \int_{-\infty}^0 \frac{(-y)^3}{\Gamma(4)} z_y dy + \int_{-\infty}^0 \frac{(-y)^4}{\Gamma(5)} z_y dy \\ &\quad + \int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \frac{t^4}{4!} D^{-1} z_0 + \frac{t^3}{3!} D^{-2} z_0 + \frac{t^2}{2!} D^{-3} z_0 + t D^{-4} z_0 + D^{-5} z_0 + \int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \frac{t^4}{4!} D^4 x_0 + \frac{t^3}{3!} D^3 x_0 + \frac{t^2}{2!} D^2 x_0 + t D^1 x_0 + x_0 + \int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy \\ &= \frac{t^4}{4!} \frac{d^4 x_0}{dt^4} + \frac{t^3}{3!} \frac{d^3 x_0}{dt^3} + \frac{t^2}{2!} \frac{d^2 x_0}{dt^2} + t \frac{dx_0}{dt} + x_0 + \int_0^t \frac{(t-y)^3}{\Gamma(4)} z_y dy \end{aligned}$$

$$\begin{array}{ll} D^{-1} z_0 = D^4 x_0 & D^5 X_t = z_t \\ D^{-2} z_0 = D^3 x_0 & D^4 X_t = D^{-1} z_t \\ \text{where } D^{-3} z_0 = D^2 x_0 \text{ since} & D^3 X_t = D^{-2} z_t \\ D^{-4} z_0 = D x_0 & D^2 X_t = D^{-3} z_t \\ D^{-5} z_0 = x_0 & DX_t = D^{-4} z_t \\ & X_t = D^{-5} z_t \end{array}$$

for which it can again be seen that the forward memory [which here is extended Markov to order 5] is a Maclaurin series,

$$\left( \frac{1}{4!} t^4 D^4 + \frac{1}{3!} t^3 D^3 + \frac{1}{2!} t^2 D^2 + \frac{1}{1!} t^1 D + 1 \right) x_0$$

and the term for the forward transition is the remainder

$$\int_0^t \frac{(t-y)^4}{\Gamma(5)} z_y dy$$

For arbitrary  $n$ , the differential form of the diffusion equation is  $D^n X_t = z_t$  which resolves for

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

Noting that since  $D^n X_t = z_t$  then  $D^{n-j} X_t = D^{-j} z_t$  and  $D^j X_t = D^{-(n-j)} z_t$  so that  $D^{-(n-j)} z_0 = D^j x_0 = (d/dt)^j x_0$  for the memory terms at the origin. Splitting the resolved variable into memory and forward transition at the origin (which gives the Maclaurin series)

$$X_t = \int_{-\infty}^0 \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy$$

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

and expanding the binomial term  $(t-y)^{n-1} = \sum_{j=0}^{n-1} {}^{n-1}C_j t^j (-y)^{n-j-1}$  in the memory component and noting that  ${}^{n-1}C_j = (n-1)! / j!(n-j-1)! = \Gamma(n) / j!\Gamma(n-j)$

$$\begin{aligned}
 X_t &= \int_{-\infty}^0 \frac{\sum_{j=0}^{n-1} {}^{n-1}C_j t^j (-y)^{n-j-1}}{\Gamma(n)} z_y dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \int_{-\infty}^0 \sum_{j=0}^{n-1} \frac{\Gamma(n)}{j!\Gamma(n-j)} \frac{t^j (-y)^{n-j-1}}{\Gamma(n)} z_y dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \int_{-\infty}^0 \sum_{j=0}^{n-1} \frac{t^j (-y)^{n-j-1}}{j!\Gamma(n-j)} z_y dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \sum_{j=0}^{n-1} \frac{t^j}{j!} \left( \int_{-\infty}^0 \frac{(-y)^{n-j-1}}{\Gamma(n-j)} z_y dy \right) + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \sum_{j=0}^{n-1} \frac{t^j}{j!} (D^{-(n-j)} z_0) + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \sum_{j=0}^{n-1} \frac{t^j}{j!} (D^j x_0) + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\
 &= \sum_{j=0}^{n-1} \frac{t^j}{j!} \left( \frac{d^j}{dt^j} x_0 \right) + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy
 \end{aligned}$$

Letting  $n$  go to infinity  $n \rightarrow \infty$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} X_t &= \lim_{n \rightarrow \infty} \left( \sum_{j=0}^{n-1} \frac{t^j}{j!} (D^j x_0) + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \right) \\
 &= \sum_{j=0}^{\infty} \frac{t^j}{j!} D^j x_0
 \end{aligned}$$

because factorials go to infinity faster than powers so that the forward transition [remainder] becomes negligible

$$\lim_{n \rightarrow \infty} \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy = 0$$

and the resolved variable is composed entirely of backward diffusion variables that reconstruct the diffusion variable's dynamics at the origin [the backward time variable]. The resolved variable for an infinite number of diffusion unit roots (an infinite number of unit roots on the diffusion side of the diffusion equation) is hence a Maclaurin series

$$\begin{aligned}
 X_t &= x_0 + \frac{1}{1!} t^1 D x_0 + \frac{1}{2!} t^2 D^2 x_0 + \frac{1}{3!} t^3 D^3 x_0 + \frac{1}{4!} t^4 D^4 x_0 + \frac{1}{5!} t^5 D^5 x_0 + \dots \\
 &= \sum_{j=0}^{\infty} \frac{t^j}{j!} D^j x_0
 \end{aligned}$$

for which is the purpose of this document. This then shows that Maclaurin series (and Taylor series for a backward time variable which partitions past and future other than the origin [time zero,  $t_0 = 0$ ]) builds up a function, within the radius of convergence, from an infinite number of homogenous unit roots [these are harmonic in the case of Fourier series]. The dynamics of the function is then specified by matching the function's derivatives at the origin [or more generally at the backward time variable for Taylor series]. Fourier's series and theorem is then the same principle but whereby the function is reconstructed from an infinite number of harmonic roots, namely a pure tone plus an infinite number of overtones which I will be showing in the next document [I have yet to type it up **(I have predicted more theory for which I am yet to develop and hence have not yet sent this declared document)**].

In more familiar notation for the Maclaurin expansion, rather than the notation of the diffusion equation and notation of initial conditions of differential equations [which are diffusion equations on  $R$ ], the diffusion equation is a function of the parameter of dynamics (time)  $X_t = f(t)$  and derivatives may then be written as  $D^j X_t = f^{(j)}(t)$  so that  $D^j x_0 = f^{(j)}(0)$  which expresses the resolved variable as

$$f(t) = f(0) + t f'(0) + \frac{t^2}{2!} f''(0) + \frac{t^3}{3!} f'''(0) + \dots = \sum_{j=0}^{\infty} \frac{t^j}{j!} f^{(j)}(0)$$

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

which will be familiar to all [mathematicians] who are reading this document. For a non-zero backward time variable (which is the origin  $t_0 = 0$  for Maclaurin series),  $t_0$ , which means splitting the memory and the forward transition at  $t_0$  rather than the origin, then gives the series (and determines where to condition for the backward variables [changing their case from large caps to small since they are then on  $R$  rather than  $S$ ])

$$X_t = \sum_{j=0}^{\infty} \frac{(t-t_0)^j}{j!} \left( \frac{d^j}{dt^j} x_t \Big|_{t=t_0} \right)$$

which is a Taylor series. Labeling the backward time variable  $t_0$  with the letter  $a$ , in familiar notation, the series can be written as

$$f(t) = f(a) + (t-a) f'(a) + \frac{(t-a)^2}{2!} f''(a) + \frac{(t-a)^3}{3!} f'''(a) + \dots = \sum_{j=0}^{\infty} \frac{(t-a)^j}{j!} f^{(j)}(a)$$

which again will be recognized at once by all who is reading this and has demonstrated the point which I set out to demonstrate, namely that Taylor and Maclaurin series are particular cases of the diffusion equation, in this case that for an infinite number of unit roots  $p(D) = D^n$  ( $n \rightarrow \infty$ ). This is then essentially an infinite order differential equation with an infinite number of initial conditions which are the derivatives that reconstruct the function. I will also demonstrate this point [when I get around to typing the results up] for Fourier's Theorem and Fourier expansions. I anticipate that the diffusion equation, as the calculus extended from  $R$  to  $S$ , will unify all of mathematics. It formally and explicitly gives the calculus a basis [which has long been recognized in linear algebra] which has been necessary for the extension from  $R$  to  $S$ . It has been overwhelmingly clear that the calculus has a basis, and this basis is made explicitly clear with the diffusion equation [with its own explicit, but unobservable, variable  $z$ ].

### Some further notes

Note that the mathematical method of obtaining the Maclaurin or Taylor coefficients by successive differentiating a power series is essentially differentiating the integral form of the diffusion equation back into the starting differential form which has been resolved to give the series, that is, successively differentiating

$$p^{-1}(D) = \int_{-\infty}^t \int_{-\infty}^t \dots \int_{-\infty}^t (\cdot) dt_1 \dots dt_{n-1} dt_n \quad (\text{for } n \rightarrow \infty)$$

back into

$$p(D) = D^n \quad (\text{for } n \rightarrow \infty)$$

and consequently converting  $X_t = D^{-n} z_t$  back to  $D^n X_t = z_t$ , that is, guessing that  $X_t = D^{-n} z_t$  is a power series and differentiating back to  $D^n X_t = z_t$  to get the coefficients. This is why it works and has been so immensely useful to mathematicians, namely, because of the action of reconstructing a function by matching its dynamics to an infinite number of resolved unit roots which then essentially provides the complete differential dynamics at the backward time variable.

The fact that this series arises from adding an infinite number of unit roots to the diffusion side of the diffusion equation [and then resolving] is also the reason for the condition on Maclaurin series that the function must be infinitely differentiable. The infinite number of  $D$ 's are the derivatives that build up the function's dynamics [the dynamics of  $X$  at the backward time variable], hence being the required condition for the technique to work since this is what it is [the solution to an infinite order differential equation, and diffusion equation when extended from  $R$  to  $S$ ].

### The remainder

The remainder term for a  $n^{\text{th}}$  order power series expansion is

$$\begin{aligned} R_n(t) &= \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy \\ &= \int_0^t \frac{u^{n-1}}{\Gamma(n)} z_{t-u} du \end{aligned}$$

which can be expressed in terms of the diffusion variable

$$\begin{aligned} R_n(t) &= \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} (D^n X_y) dy \\ &= \int_0^t \frac{u^{n-1}}{\Gamma(n)} (D^n X_{t-u}) du \end{aligned}$$

which results in the Lagrange form for the remainder when the mean value theorem is applied to average out the integral – this is essentially the mean of the forward transition. With the extension from  $R$  to  $S$  with the stochastic basis, there is then **Aaron's form for the remainder** [or alternatively, the **mean-variance form of the remainder**]. Remembering (by definition) that the basis is normal with mean  $\mu$  and variance  $\sigma^2/dt$

$$z_t \sim N(\mu, \sigma^2/dt)$$

so the remainder is statistical. The mean of the remainder is

$$\begin{aligned} E(R_n(t)) &= E\left(X_t - \sum_{j=0}^{n-1} \frac{1}{j!} t^j D^j X_0 \mid X_0 = x_0, DX_0 = DX_0, D^2 X_0 = D^2 x_0, \dots, D^{n-1} X_0 = D^{n-1} x_0\right) \\ &= E\left(\int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy\right) = \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} E(z_y) dy = \mu \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} dy \\ &= \mu(t^n/n!) \end{aligned}$$

which is the Lagrange form of the remainder and the mean value of the integral taking the basis as a constant (which just assumes that it is all mean ( $\sigma^2 = 0$ )) which is true for functions on  $R$  but not on  $S$ ). The variance is

$$\begin{aligned} Var(R_n(t)) &= Var\left(X_t - \sum_{j=0}^{n-1} \frac{1}{j!} t^j D^j X_0 \mid X_0 = x_0, DX_0 = DX_0, D^2 X_0 = D^2 x_0, \dots, D^{n-1} X_0 = D^{n-1} x_0\right) \\ &= Var\left(\int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy\right) = \int_0^t Var\left(\frac{(t-y)^{n-1}}{\Gamma(n)} z_y dy\right) = \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} Var(z_y) (dy)^2 \\ &= \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} \frac{\sigma^2}{dy} (dy)^2 = \sigma^2 \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} dy \\ &= \frac{\sigma^2 t^{2n-1}}{(2n-1)((n-1)!)^2} \end{aligned}$$

that is

$$R_n(t) \sim N\left(\mu(t^n/n!), \sigma^2 t^{2n-1}/(2n-1)((n-1)!)^2\right)$$

which is normally distributed and is essentially the forward transition of the  $n$ th order integrated Brownian motion that reconstructs the function with the  $n$  repeated unit roots. The full density of the remainder is then Gaussian with these transition cumulants [which is one of my many mathematical discoveries (original equations): I was mathematically humble until experiencing the idiot PhD examiner at which time it became overwhelmingly apparent that I had to back (bat for) myself or I would not get credit for anything]. The statistical 95% confidence limits are then  $E(R_n(t)) \pm 1.96 Var(R_n(t))$  as per standard statistical theory [upon which the stochastic calculus is based, by union with the calculus]. With a backward time variable other than the origin, i.e., for Taylor rather than Maclaurin series, the remainder term is

$$R_n(t-t_0) \sim N\left(\mu((t-t_0)^n/n!), \sigma^2 (t-t_0)^{2n-1}/(2n-1)((n-1)!)^2\right)$$

These distributions then allow the power series expansions to be simulated on  $S$  for stochastic variables. Prudence is required for choosing  $\mu$  and  $\sigma^2$  because the variance grows quickly with the repeated integrations.

### Reconstructing a function from moderated roots: an extra degree of freedom

Instead of generating the power series from an infinite number of unit roots ( $p(D) = D^n$  for  $n \rightarrow \infty$ ), the power series can be generated with homogenous moderated roots ( $p(D) = (D+\alpha)^n$  for  $n \rightarrow \infty$ ), allowing for an extra degree of freedom (that of the moderating parameter) to the technique of Taylor and Maclaurin. Although I am doing this just for the sake of mathematics, it may allow for a

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

wider class of functions and will definitely be useful in ways yet to be seen. I definitely know that it will be useful in one way or another.

Taking the exact same approach as that for all unit roots at the start of this document and starting again with the diffusion equation relating the diffusion variable to the basis variable

$$X_t = z_t$$

In the case now, we will be considering all moderated roots [which have imaginary harmonics if paired up in hyperbolic pairs but this will not be considered here]; adding the  $n$  moderated roots  $p(D) = (D + \alpha)^n$  to the diffusion side [again with no inverse diffusion  $q(D) = D^0 = 1$ ], the diffusion equation becomes

$$(D + \alpha)^n X_t = z_t \quad \left( \frac{d}{dt} + \alpha \right)^n X_t = z_t \quad \sum_{j=0}^n {}^nC_j \alpha^{n-j} (D^j X_t) = z_t$$

Resolving

$$X_t = (D + \alpha)^{-n} z_t \\ = e^{\alpha t_n} \int_{-\infty}^{t_n} e^{-\alpha t_n} e^{\alpha t_{n-1}} \int_{-\infty}^{t_{n-1}} e^{-\alpha t_{n-1}} \dots e^{\alpha t_1} \int_{-\infty}^{t_1} e^{-\alpha t_1} z_t dt_1 \dots dt_{n-1} dt_n$$

which by similar methods as the unit roots can be expressed as

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

(where  $t = t_1$ ) and is the formula for the resolved form of the diffusion equation for an arbitrary order  $n$  of repeated moderated roots. This is that utilized here to generalize Taylor and Maclaurin, allowed for by constructing the expansions in terms of the diffusion equation, for purposes yet to be seen and as yet unknown.

**The proof of this is similar to that of unit roots whereby the order of integration is reversed and the inner integrals evaluated. This was the main theory of BOOK 3 of my PhD meaning that any diffusion equation of arbitrary order can be evaluated with a single integration [a PhD which should have been passed outright]. Seeing that the mathematics is similar, instead of a proof with similar mathematics, I will just show a way in which this concise result can be seen; writing the diffusion side dynamics in exponential form**

$$p(D) = (D + \alpha)^n = (e^{-\alpha t} D e^{\alpha t})^n = \prod_{j=1}^n e^{-\alpha t} D e^{\alpha t} = e^{-\alpha t} D e^{\alpha t} \cdot e^{-\alpha t} D e^{\alpha t} \dots e^{-\alpha t} D e^{\alpha t} \cdot e^{-\alpha t} D e^{\alpha t} \\ = e^{-\alpha t} D^n e^{\alpha t}$$

Hence,  $(D + \alpha)^n X_t = z_t$  can be expressed as  $e^{-\alpha t} D^n e^{\alpha t} X_t = z_t$  so that

$$D^n e^{\alpha t} X_t = e^{\alpha t} z_t$$

and utilizing the result  $D^{-n} = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} (\cdot) dy$  for all unit roots  $p(D) = D^n$  to resolve the  $D^n$

$$e^{\alpha t} X_t = D^{-n} e^{\alpha t} z_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{\alpha y} z_y dy$$

that is

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

which is the fully resolved equation [prior to splitting into forward transition and memory] and the result that was to be shown. There are several other ways of showing this as well.

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

Instead of going through the mathematics for  $n=1, 2, 3, 4$ , and 5 and then arbitrary  $n$ , I will just do the mathematics for arbitrary  $n$  since this is similar, although slightly more complicated  $p(D) = (D + \alpha)^n$ , than that for all unit roots  $p(D) = D^n$ . The mean-variance (Aaron's) form of the remainder term is on the other hand quite a bit more complicated with the decay constant but this involves Poisson probabilities so those with actuarial degrees (such as myself) will probably feel comfortable with this [Poisson probabilities and the Poisson process (and the soon to be recognized<sup>1</sup> Aaron's lemma, the equivalent to Ito's lemma for Poisson rather than Gaussian diffusion)].

From the resolved form of the diffusion equation  $(D + \alpha)^n X_t = z_t$  for  $n$  repeated moderated roots

$$X_t = \int_{-\infty}^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

Noting that from  $(D + \alpha)^n X_t = z_t$  also  $(D + \alpha)^{n-j} X_t = (D + \alpha)^j z_t$  and  $(D + \alpha)^{-j} X_t = (D + \alpha)^{-(n-j)} z_t$  as well as  $X_t = (D + \alpha)^{-n} z_t$  so that the backward diffusion variables that comprise the roots (derivatives) that reconstruct the function at the backward (time) variable are  $(D + \alpha)^j z_0 = (D + \alpha)^{n-j} x_0$  and  $(D + \alpha)^{-(n-j)} z_0 = (D + \alpha)^{-j} x_0$  as well as the familiar initial condition of first order (Markov) equations  $(D + \alpha)^{-n} z_0 = x_0$  (here arising from an equation with the extended Markov property to order  $n$ ). Splitting the resolved variable into memory and forward transition (by conditioning) at the backward time variable

$$X_t = \int_{-\infty}^0 \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

for which the forward transition becomes the remainder term when seen in the context of the power series of backward variables as a formal Maclaurin expansion. Taking the same approach as for the unit root derivatives, and noting that

$$\frac{(t-y)^{n-1}}{\Gamma(n)} = \frac{(t+(-y))^{n-1}}{\Gamma(n)} = \frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} {}^{n-1}C_j t^j (-y)^{n-j-1} = \frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{\Gamma(n)}{j! \Gamma(n-j)} t^j (-y)^{n-j-1} = \sum_{j=0}^{n-1} \frac{t^j (-y)^{n-j-1}}{j! \Gamma(n-j)}$$

and expanding the memory term so as to state it in terms of diffusion terms rather than the basis variable

$$\begin{aligned} X_t &= \int_{-\infty}^0 z_y e^{-\alpha(t-y)} \sum_{j=0}^{n-1} \frac{t^j (-y)^{n-j-1}}{j! \Gamma(n-j)} dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy \\ &= e^{-\alpha t} \sum_{j=0}^{n-1} \frac{t^j}{j!} \int_{-\infty}^0 \frac{(-y)^{n-j-1}}{\Gamma(n-j)} z_y e^{-\alpha(-y)} dy + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy \\ &= e^{-\alpha t} \sum_{j=0}^{n-1} \frac{t^j}{j!} (D + \alpha)^{n-j-1} z_0 + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy \\ &= e^{-\alpha t} \sum_{j=0}^{n-1} \frac{t^j}{j!} (D + \alpha)^j x_0 + \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy \end{aligned}$$

which is a generalized Maclaurin Series to  $n$  terms

$$X_t = e^{-\alpha t} \sum_{j=0}^{n-1} \frac{t^j}{j!} (D + \alpha)^j x_0 + R_n(t; \alpha)$$

with remainder term

$$\int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

Letting  $n$  go to infinity  $n \rightarrow \infty$

---

<sup>1</sup> Required for the complete statistical elucidation of Thorium ( $4n$ ), Neptunium ( $4n+1$ ), Radium ( $4n+2$ ), and Actinide ( $4n+3$ ) radioactive decay series [which will show that this is also the calculus extended from  $R$  to  $S$ ].

$$X_t = e^{-\alpha t} \sum_{j=0}^{\infty} \frac{t^j}{j!} (D + \alpha)^j x_0$$

$$= e^{-\alpha t} \left( 1 + t(D + \alpha) + \frac{1}{2!} t^2 (D + \alpha)^2 + \frac{1}{3!} t^3 (D + \alpha)^3 + \frac{1}{4!} t^4 (D + \alpha)^4 + \frac{1}{5!} t^5 (D + \alpha)^5 + \dots \right) x_0$$

and the equivalent for the Taylor series is applied similarly as for unit roots by splitting at  $t_0$  rather than the origin. The decay constant  $\alpha$  in the derivatives increase the reconstructing derivatives in the series and the exponential moderator at the front then scales them back down giving more overall dynamic to the expansion from an extra degree of freedom. This is essentially a polynomial in **Newton's law of Cooling**, that is, a Maclaurin/Taylor series [depending on where the transition is split into memory and forward transition] in powers of the **Law of Cooling**.

Note that

$$X_t = e^{-\alpha t} \sum_{j=0}^{\infty} \frac{t^j}{j!} (D + \alpha)^j x_0 = e^{-\alpha t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{k=0}^j \frac{j!}{k!(j-k)!} D^k \alpha^{j-k} x_0 = e^{-\alpha t} \sum_{j=0}^{\infty} t^j \sum_{k=0}^j \frac{1}{k!(j-k)!} D^k \alpha^{j-k} x_0$$

$$= e^{-\alpha t} \sum_{k=0}^{\infty} (D^k x_0) \sum_{j=k}^{\infty} \frac{t^j \alpha^{j-k}}{k!(j-k)!}$$

which is a unit root series weighted by sums of Bessel functions, theory for which I find very interesting and which I may elaborate on at some time in the future. Anyway, the remainder term is now considered. In particular, the mean-variance form of the remainder.

### Remainder for the generalized series

Similarly as the Taylor and Maclaurin series considered for the extension from  $R$  to  $S$  for all unit root derivatives, the remainder term is then stochastic (statistical) for any  $n < \infty$ . Because of the exponential term, these aren't all that easily integrated (as simple powers) for determining the mean and variance from the integration of the basis function. Instead, these are gamma integrals and are therefore expressed once evaluated by repeated parts [and here by recursion] as incomplete gamma integrals. This means that the cumulants (mean and variance for a normal basis) are then probabilities of a Poisson process [and are hence analytic to the order of terms in the finite sum for any  $n < \infty$ ], such as that for counting car accidents in a motor vehicle insurance portfolio or counting clicks on a Geiger counter.

From the definition of the basis variable (orthogonal normal and hence a Gaussian-Hilbert vector space), which is orthogonal normal with mean  $\mu$  and variance  $\sigma^2 / dt$

$$z_t \sim N(\mu, \sigma^2 / dt)$$

which then generates the statistical variation in the remainder term

$$R_n(t; \alpha) = \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy$$

for non-negligible  $\sigma^2$  so the remainder is statistical with both mean and variance due to the basis. The mean of the remainder is then

$$E(R_n(t; \alpha)) = E\left(X_t - e^{-\alpha t} \sum_{j=0}^{n-1} \frac{1}{j!} t^j (D + \alpha)^j x_0 \mid X_0 = x_0, DX_0 = Dx_0, D^2 X_0 = D^2 x_0, \dots, D^{n-1} X_0 = D^{n-1} x_0\right)$$

$$= E\left(\int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy\right) = \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} E(z_y) dy = \mu \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} dy$$

$$= \frac{\mu}{\alpha^n} \left(1 - \sum_{j=0}^{n-1} (\alpha t)^j e^{-\alpha t} / j!\right)$$

$$= \frac{\mu}{\alpha^n} \left(1 - \frac{\Gamma(n; \alpha t)}{\Gamma(n)}\right)$$

where  $\Gamma(n; \alpha t)$  is the incomplete gamma function. Note the filtration at the backward variables (the conditioned variables at the backward time variable and the order of the extended Markov property which summarizes the past of the diffusion variable) is still

$$X_0 = x_0, DX_0 = Dx_0, D^2 X_0 = D^2 x_0, \dots, D^{n-1} X_0 = D^{n-1} x_0$$

## Aaron Lloyd-Jones: Maclaurin and Taylor Series from the Diffusion Equation

which includes all  $n-1$  backward derivatives (including  $D^0 x_0$ ) so that the sequence includes all of the conditioned variables at the backward time variable and therefore the filtration sequence is not required to be expressed as say

$$X_0 = x_0, (D + \alpha)X_0 = (D + \alpha)^2 x_0, (D + \alpha)^2 X_0 = (D + \alpha)^3 x_0, \dots, (D + \alpha)^{n-1} X_0 = (D + \alpha)^n x_0$$

A bit pedantic, but I nevertheless think that it is important that it is notionally understood. The filtration is the diffusion variable's memory, and once accumulated with the powers of  $t$  (and the exponential moderator here), it is the forward memory [at the forward time variable]. The integral was evaluated with an application of parts to give the recursion for the integral  $I_n$

$$I_n = \frac{I_{n-1}}{\alpha} - \frac{t^{n-1} e^{-\alpha t}}{\alpha \Gamma(n)} \quad \text{where } I_n = \int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} dy \text{ and } I_1 = \frac{1 - e^{-\alpha t}}{\alpha}$$

(rather than repeating the parts), which generates the Poisson probabilities, and then solved for  $I_n$ . The fact that these terms are probabilities of a Poisson process  $N_t \sim P(\alpha t)$  is no coincidence and will be elaborated on at a later time. Similarly as for unit roots, the mean of the forward transition is the mean value of the remainder term and hence the Lagrange form (for the generalized expansion) taking the basis  $z$  as a constant  $\mu$ . For the extension to  $S$ , the variance is

$$\begin{aligned} \text{Var}(R_n(t; \alpha)) &= \text{Var}\left(X_t - e^{-\alpha t} \sum_{j=0}^{n-1} \frac{1}{j!} t^j (D + \alpha)^j x_0 \mid X_0 = x_0, DX_0 = Dx_0, D^2 X_0 = D^2 x_0, \dots, D^{n-1} X_0 = D^{n-1} x_0\right) \\ &= \text{Var}\left(\int_0^t \frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy\right) = \int_0^t \text{Var}\left(\frac{(t-y)^{n-1}}{\Gamma(n)} e^{-\alpha(t-y)} z_y dy\right) = \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} e^{-2\alpha(t-y)} \text{Var}(z_y) (dy)^2 \\ &= \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} e^{-2\alpha(t-y)} \frac{\sigma^2}{dy} (dy)^2 = \sigma^2 \int_0^t \frac{(t-y)^{2n-2}}{\Gamma(n)^2} e^{-2\alpha(t-y)} dy \\ &= \frac{\sigma^2}{(2\alpha)^{2n-1}} \frac{B(n, n)^{-1}}{(2n-1)} \left(1 - \sum_{j=0}^{2n-2} (2\alpha t)^j e^{-2\alpha t} / j!\right) \\ &= \frac{\sigma^2}{(2\alpha)^{2n-1}} \frac{B(n, n)^{-1}}{(2n, 1)} \left(1 - \frac{\Gamma(2n-1; 2\alpha t)}{\Gamma(2n-1)}\right) \end{aligned}$$

where  $B(n, n) = \int_0^1 t^{n-1} (1-t)^{n-1} dt = \Gamma(n)^2 / \Gamma(2n)$  is a symmetric beta function ( $B(n, n)^{-1} = \Gamma(2n) / \Gamma(n)^2$ ) and  $\Gamma(2n-1; 2\alpha t)$  is an incomplete gamma function. In a similar (but not identical) method as the mean, the (scaled) integral [since the squared gamma functions don't stepwise cancel] was evaluated with an application of parts to give the recursion

$$I_n = \frac{2n-2}{2\alpha} I_{n-1/2} - \frac{t^{2n-2} e^{-2\alpha t}}{2\alpha} \quad \text{where } I_n = \int_0^t (t-y)^{2n-2} e^{-2\alpha(t-y)} dy \text{ and } I_1 = \frac{1 - e^{-2\alpha t}}{2\alpha}$$

where the  $2n-1$  terms in the sum are probabilities from the Poisson process  $N_t \sim P(2\alpha t)$ , i.e., that for the mean with twice the rate intensity. The [Gaussian] distribution of the remainder for the generalized Maclaurin expansion to  $n$  terms is then

$$R_n(t; \alpha) \sim N\left(\frac{\mu}{\alpha^n} \left(1 - \sum_{j=0}^{n-1} (\alpha t)^j e^{-\alpha t} / j!\right), \frac{\sigma^2}{(2\alpha)^{2n-1}} \frac{B(n, n)^{-1}}{(2n-1)} \left(1 - \sum_{j=0}^{2n-2} (2\alpha t)^j e^{-2\alpha t} / j!\right)\right)$$

which can then allow for simulations and confidence limits for data.

I have thus proved what I set out to demonstrate, that Taylor and Maclaurin series are particular examples of the diffusion equation, and have used the diffusion equation to give these techniques a formal basis [that from the diffusion equation] and extended these techniques for the mean-variance form of the remainder and for a non trivial moderating decay constant  $\alpha$ . In the next document I send, I will be doing the same for Fourier series and synthesis.

## References

Borowski, E. J. and Borwein, J. M., (1989) *Collins Dictionary: Mathematics*. HarperCollins.

Lloyd-Jones, A., (2011) *Stochastic Calculus and Option Valuation: A Perspective from Mathematical Statistics*, Ph. D. Dissertation. Discipline of Finance, the University of Sydney.